APPLICATION OF THE METHOD OF SUCCESSIVE APPROXIMATIONS TO THE NONLINEAR EVOLUTION OF TURBULENT PERTURBATIONS

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The method of successive approximations is used to analyze the behavior of the energy of two-dimensional turbulent perturbations in a flow with a constant average-velocity gradient.

A continuous supply of energy is required to maintain turbulence in the flow of a viscous, incompressible liquid. This energy is supplied by the work performed by the average flow in opposition to the Reynolds stresses. If no energy is supplied, the turbulence degenerates. The turbulence develops in regions with a nonvanishing gradient in the average velocity, so that it is important to study the behavior of the energy of turbulent perturbations in such a flow. It would then be possible to evaluate the stability of the flow against perturbations.

Let us examine the development of turbulent perturbations in the plane-parallel flow of a viscous, incompressible liquid in which the average flow velocity is $\vec{U} = (\beta x_2, 0, 0)$, where β = const.

In accordance with the results of [1], we restrict this analysis to two-dimensional perturbations.

The energy of the turbulent perturbations in a flow with a constant average-velocity gradient increases as time elapses, reaches a maximum (governed by the Reynolds number and the initial amplitude), and then slowly decays, according to the linear theory [2], which holds for short time intervals. If the initial amplitude is sufficiently high, the turbulent perturbations can reach a maximum energy beyond the scope of the linear approximation for fluctuations of the velocity $\vec{u}(\vec{x}, t)$, so that the terms in the original equations which are nonlinear in $\vec{u}(\vec{x}, t)$ must be taken into account.

To solve the nonlinear problem we can use an infinite system of coupled moment equations; since this system is not closed, certain additional assumptions must be made regarding the coupling of the higher-order moments in order to close the system. If for this purpose we use the assumption that the semiinvariants of the velocity of a fixed order $n + 1 \geq 4$ vanish for both $n = 3$ (this is the Millionshchikov assumption [3], which permits the fourth-order moments to be expressed in terms of the second-order moments) and $n = 4$ (the fifth-order moments are expressed in terms of special combinations of the second- and third-order moments), then we find that the energy of the turbulent perturbations becomes negative [4].

For a flow with a constant average-velocity gradient we can make direct use of the Navier–Stokes equation for the velocity fluctuations, along with the system of moment equations. This approach is more accurate in the sense that it does not rest on any assumptions equivalent to closing assumptions. The system of nonlinear equations obtained as a result can be solved by the method of successive approximations. Using this method, Khazen [5] found estimates for each of the approximations and specified some of their properties.

In the present paper we attempt to extend the method of successive approximations to the problem of the nonlinear evolution of turbulent fluctuations in a flow with a constant

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average-velocity gradient, and we discuss the results found in a numerical solution.

The velocity fluctuations $u_i(\vec{x}, t)$ in this type of flow satisfy the equation [5]

$$\frac{\partial u_i}{\partial t} + U_1 \frac{\partial u_i}{\partial x_1} + \beta\delta_{i1}u_2 + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k}. \quad (1)$$

Introducing the Fourier transform of the velocity fluctuation,

$$v_i(\vec{k}, t) = \int_{-\infty}^{\infty} u_i(\vec{x}, t)\exp(i\vec{k}\vec{x})\, d\vec{x}$$

and using the incompressibility equation in order to eliminate the pressure fluctuations and in order to introduce the function $(\vec{k}, t)$ such that

$$v_i(\vec{k}, t) = (\delta_{i2}k_1 - \delta_{i1}k_2)\varphi(\vec{k}, t),$$

we reduce the problem to the solution of an equation for the function $(\vec{k}, t)$ [5]:

$$L\varphi(\vec{k}, t) = \frac{i}{k^2}\int_{-\infty}^{\infty} f(\vec{k}, \vec{k}')\varphi(\vec{k}', t)\varphi(\vec{k}-\vec{k}', t)\, d\vec{k}' \quad (2)$$

with the initial condition

$$\varphi(\vec{k}, 0) = A\exp(-k^2)\sum_{m,n}[\delta(\vec{k}-\vec{k}^0_{mn}) + \delta(\vec{k}+\vec{k}^0_{mn})], \quad (3)$$

which corresponds to a system of eddy perturbations.

Here and below,

$$L = \frac{\partial}{\partial t} - k_1\frac{\partial}{\partial k_2} - 2\frac{k_1k_2}{k^2} + \frac{k^2}{\mathrm{Re}},$$

$$f(\vec{k}, \vec{k}') = (k_1k_1' + k_2k_2')(k_2k_1' - k_1k_2'),$$

and we are using the dimensionless variables $\tilde{k}_i = k_i\alpha^{-1}$, $\tilde{t} = t\beta$, and $\tilde{A} = A\alpha\beta^{-1}$ (below we omit the tilde).

In accordance with [5], we see a solution of Eq. (2) by the method of successive approximations:

$$\varphi(\vec{k}, t) = \sum_{r=1}^{\infty}\varphi_r(\vec{k}, t),$$

where

$$\varphi_r(\vec{k}, t) = D_r(\vec{k}, t) + iB_r(\vec{k}, t).$$

In the first, linear, approximation (r = 1), we find equations for the functions $D_1(\vec{k}, t)$ and $B_1(\vec{k}, t)$:

$$LD_1(\vec{k}, t) = 0, \quad LB_1(\vec{k}, t) = 0 \quad (4)$$

with initial conditions which follow from (3),

$$D_1(\vec{k}, 0) = A\exp(-k^2)\sum_{m,n}[\delta(\vec{k}-\vec{k}^0_{mn}) + \delta(\vec{k}+\vec{k}^0_{mn})],$$

$$B_1(\vec{k}, 0) = 0.$$

Using the expressions for the first integrals

$$k_i^0 = k_i + \delta_{i2}k_1^0 t, \quad (5)$$

which are found from the characteristic system for Eqs. (4), we can write the solution of the first-approximation equations as [5]

$$D_1(\vec{k}, t) = A\frac{k_1^2 + (k_2 + k_1t)^2}{k^2}\exp[-k_1^2 - (k_2 + k_1t)^2]\exp\left[-\frac{1}{\mathrm{Re}}\left(k^2t + k_1k_2t^2 + \frac{1}{3}k_1^2t^3\right)\right], \quad B_1(\vec{k}, t) = 0. \quad (6)$$

For $r \geq 2$, the functions $D_r(\vec{k}, t)$ and $B_r(\vec{k}, t)$ satisfy the system of equations

$$L_1 D_r(\vec{k}, t) = -\frac{1}{k^2}\int_{-\infty}^{\infty} f(\vec{k}, \vec{k}')\left\{\sum_{p+q=r}[D_p(\vec{k}', t) B_q(\vec{k}-\vec{k}', t) + B_q(\vec{k}', t) D_p(\vec{k}-\vec{k}', t)]\right\} d\vec{k}', \quad (7)$$

$$L_1 B_r(\vec{k}, t) = \frac{1}{k^2}\int_{-\infty}^{\infty} f(\vec{k}, \vec{k}')\left\{\sum_{p+q=r}[D_p(\vec{k}', t) D_q(\vec{k}-\vec{k}', t) - B_p(\vec{k}', t) B_q(\vec{k}-\vec{k}', t)]\right\} d\vec{k}', \quad (8)$$

where

$$L_1 = \frac{d}{dt} - 2\frac{k_1 k_2}{k^2} + \frac{k^2}{\mathrm{Re}}$$

is an operator.

The initial conditions for the functions $D_r(\vec{k}, t)$ and $B_r(\vec{k}, t)$ are

$$D_r(\vec{k}, 0) = B_r(\vec{k}, 0) = 0 \quad (r \geqslant 2), \quad (9)$$

as follows from (3).

The evolution of all the perturbations cannot be followed in the numerical calculations, so we consider a finite number of these perturbations. We replace the integrals on the right sides of Eqs. (7) and (8) by finite sums, and we reduce the problem to the solution of a system of inhomogeneous second-order equations for the functions of the r-th approximation, $D_r \cdot (\vec{k}, t)$ and $B_r(\vec{k}, t)$.

System (7), (8), in which the integrals on the right sides are replaced by finite sums, the initial conditions (9), and law (5) for the behavior of the components of the wave vector $\vec{k}$, was approximated by a system of finite-difference equations with the help of an explicit-difference scheme. This system was solved numerically. Functions of the first approximation $D_1(\vec{k}, t)$ and $B_1(\vec{k}, t)$ are calculated in accordance with (6).

The results calculated for Re = 10^4, A = 0.5, $-N \le n \le N$, $-M \le m \le M$, N = M = 4, and $\vec{k}^* = (0.02; 0.1)$ are shown in Fig. 1; here the curves show the ratio of the perturbation energy at time t,

$$E_r(t) = \sum_{s=1}^{r}\int_{-\infty}^{\infty} k^2 \varphi_s(\vec{k}, t)\bar{\varphi}_s(\vec{k}, t)\, d\vec{k}$$

to the initial energy E(0) for r = 1, 2, ..., 6 ($\bar{\varphi}$ is the complex conjugate).

In the first, linear, approximation (curve 1), the energy of the eddy perturbations decays as time elapses. There are extrema on curve 1 because this curve is a measure of the total energy of the (2N + 1) (2M + 1) − 1 noninteracting eddy perturbations with various wave numbers, which reach a maximum energy at different times, depending on the ratio nk_2^*/mk_1^* (Fig. 2).

Analysis of these results shows that, for these parameter values, there is a strong interaction between perturbations, which results in rapid energy exchange between perturbations. In turn, this exchange leads to a more rapid removal of energy from the average flow. The energy obtained from the average flow is concentrated in the individual perturbations, so that the energy of these perturbations increases without bond, as does the energy of the eddy system as a whole.

We see from Fig. 1 that the perturbation energy calculated in the third approximation (r = 3) gives a physically correct description of the nonlinear development of turbulent perturbations.

The higher-order approximations refine this solution.

The results of these calculations of the energy of turbulent perturbations by the method of successive approximations show that this method can be applied to the solution of nonlinear problems of the evolution of turbulence and give results in agreement with the available experimental data.

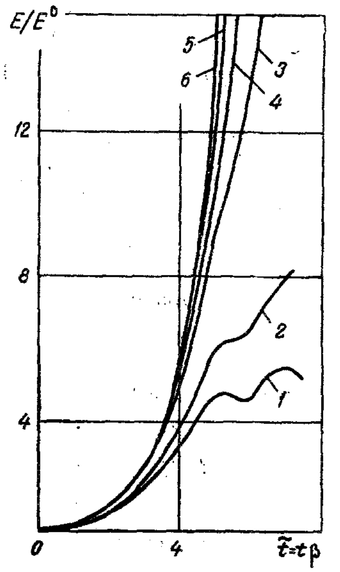

Fig. 1

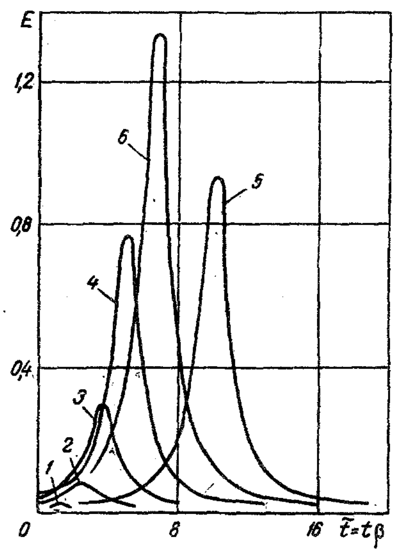

Fig. 2

Fig. 1. Time evolution of the normalized perturbation energy with various approximations r taken into account (the curves are labeled with the value of r).

Fig. 2. Time evolution of the energy of individual perturbations. 1) $\vec{k}^o_{mn} = (0.08;\ 0.1)$; 2) $\vec{k}^o_{mn} = (0.08;\ 0.2)$; 3) $\vec{k}^o_{mn} = (0.08;\ 0.3)$; 4) $\vec{k}^o_{mn} = (0.08;\ 0.4)$; 5) $\vec{k}^o_{mn} = (0.02;\ 0.2)$; 6) $\vec{k}^o_{mn} = (0.06;\ 0.4)$.

NOTATION

$\vec{u}(\vec{x}, t)$, $p(\vec{x}, t)$, perturbations of the velocity and pressure, respectively; ν, kinematic viscosity; ρ, density; α^{-1}, scale dimension of the eddies; A, initial amplitude; $Re = \beta\alpha^{-2}\nu^{-1}$, Reynolds number; $\delta(k)$, two-dimensional Dirac function; δ_{ij}, Kronecker delta; $\vec{k}^o_{mm} = (mk^*_1;\ nk^*_2)$; $k^2 = k_1^2 + k_2^2$.

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